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Singular Lagrangians with higher derivatives

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Abstract. The Hamiltonian formalism for systems with singular Lagrangians of the second order is constructed. A new method is proposed for obtaining the equations of motion in the phase space for theories with singular Lagrangians and the connection of the Lagrangian and Hamiltonian descriptions is traced in detail. As an example, a generalisation of the relativistic point action is considered. It involves both the length and curvature of the point trajectory in spacetime.

1. Introduction

Quantum field theories with Lagrangians containing derivatives of field functions higher than first order have a bad reputation because of ghost states with negative norm and, as a consequence, the possibility of unitarity violation [1]. However, such theories also have attractive properties; in particular, the convergence of the corresponding Feynman diagrams is improved. Therefore, gauge theories with higher derivatives [2-4] and gravity models with quadratic and higher-order curvature corrections to the Einstein-Hilbert action [5-12] have been considered. These theories are described by singular or degenerate Lagrangians with higher derivatives. Recently such Lagrangians have been used in some string models [11, 23].

The quantisation of the Yang-Mills fields has shown that canonical quantisation is most suitable for the investigation of unitarity properties of quantum gauge fields. This approach is based on the Hamiltonian description of classical dynamics. The Hamiltonian formalism for the usual gauge fields is constructed with the aid of the Dirac theory of the generalised Hamiltonian systems with constraints of the first order [13-16].

It is natural to explore the ghost-state problem and unitarity in theories with singular Lagrangians with higher derivatives in the framework of the canonical quantisation as well. For this purpose, however, the Hamiltonian formalism for these theories must be constructed, which is the basic aim of the present paper. A new method of obtaining the equations of motion in the phase space for theories with singular Lagrangians is proposed and the connection of the Lagrangian and Hamiltonian descriptions is traced in detail. For simplicity only the degenerate Lagrangians of second order will be considered. Recently [17] the same consideration has been repeated in terms of modern differential geometry.

This paper is organised as follows. In the § 2 the canonical variables are introduced and the definition of singular Lagrangians is given. In § 3 the equations of motion in the phase space are obtained in a new way by differentiation of the canonical

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Hamiltonian. In § 4 it is shown how one can get all the secondary constraints in the framework of the Lagrangian formalism and using the equations of motion in the Euler form. In § 5, as an example, a generalisation of the relativistic action of a point particle is considered: to the usual action proportional to the length of the world trajectory of a particle one adds the integral along this trajectory of its curvature [18]. The Hamiltonian description of the classical dynamics of this object is given and the transition to quantum theory is briefly discussed. In conclusion unsolved problems in this approach are noted.

2. The singular Lagrangians of second order

Let us consider a system with a finite number, n, of degrees of freedom. Let $x = (x_1, x_2, \ldots, x_n)$ be generalised coordinates of this system and

$$L(x, \dot{x}, \ddot{x}), \qquad \dot{x} \equiv \mathrm{d}x(t)/\mathrm{d}t \tag{2.1}$$

be its Lagrangian function. The Euler equations are

$$\frac{\partial L}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_i} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial \ddot{x}_i} = 0 \qquad i = 1, 2, \ldots, n. \tag{2.2}$$

The canonical variables for Lagrangian (2.1) are introduced in the following way:

$$q_{1i} = x_i \qquad q_{2i} = \dot{x}_i \tag{2.3}$$

$$\begin{aligned} p_{1i} &= \frac{\partial L}{\partial \dot{x}_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{x}} \\ &= \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial^2 L}{\partial \ddot{x}_i \partial x_j}\dot{x}_j - \frac{\partial^2 L}{\partial \ddot{x}_i \partial \dot{x}_j}\ddot{x}_j - \frac{\partial^2 L}{\partial \ddot{x}_i \partial \ddot{x}_j}\dddot{x}_j \end{aligned} \tag{2.4}$$

$$p_{2i} = \frac{\partial L}{\partial \ddot{x}_i} \qquad i, j = 1, 2, \ldots, n. \tag{2.5}$$

As usual, the summation over repeated indices in the corresponding limits is supposed.

Lagrangian (2.1) is called non-degenerate if the canonical variables q_1, q_2, p_1 and p_2 introduced according to (2.3)-(2.5) are independent, i.e. if there are no equation of the form†

$$f(q_1, q_2, p_1, p_2) = 0 \tag{2.6}$$

which become identities with respect to x, $\dot{x}$, $\ddot{x}$, $\dddot{x}$ after the substitution into them of definitions (2.3)-(2.5). Otherwise, i.e. when the relations (2.6) are valid, Lagrangian (2.1) is called singular or degenerate.

The condition that the Lagrangian is non-singular is obviously equivalent to the requirement that equations (2.4) and (2.5) can be solved uniquely with respect to the variables $\ddot{x}_i$ and $\dddot{x}_i$, $i = 1, \ldots, n$, in the form

$$\ddot{x}_i = \ddot{x}_i(q_1, q_2, p_2) \qquad \dddot{x}_i = \dddot{x}_i(q_1, q_2, p_1, p_2) \qquad i = 1, \ldots, n. \tag{2.7}$$

For this solution it is necessary that in the whole range of variables x, $\dot{x}$, $\ddot{x}$ the condition

$$\operatorname{rank}\|\Lambda_{ij}\| = n \tag{2.8}$$

† It is supposed that equations (2.6) do not reduce to the form $g(q_1, q_2) = 0$.

is fulfilled, where

$$\Lambda_{ij}(x,\dot{x},\ddot{x}) = \frac{\partial^2 L}{\partial \ddot{x}_i \partial \ddot{x}_j} \qquad 1 \leqslant i, j \leqslant n. \tag{2.9}$$

If condition (2.8) is satisfied, then there are no relations (2.6). To prove this, let us suppose the opposite, i.e. let the constraint (2.6) take place, with not all the derivatives $\partial f/\partial p_{1i}$, $i = 1, \ldots, n$, vanishing simultaneously. Substituting definitions (2.4) and (2.5) into (2.6) we get the identity with respect to x, $\dot{x}$, $\ddot{x}$, $\dddot{x}$. Differentiation of this identity gives

$$\frac{\partial f}{\partial p_{1k}} \frac{\partial p_{1k}}{\partial \dddot{x}_j} = -\frac{\partial f}{\partial p_{1k}} \Lambda_{kj} = 0 \tag{2.10}$$

which obviously contradicts (2.8). If the function f in (2.6) does not depend on p_1, then the derivatives $\partial f/\partial p_{2k}$, $k = 1, \ldots, n$, cannot vanish simultaneously. Differentiating (2.6) with respect to $\ddot{x}_j$ we obtain

$$\frac{\partial f}{\partial p_{2k}} \frac{\partial p_{2k}}{\partial \ddot{x}_j} = \frac{\partial f}{\partial p_{2k}} \Lambda_{kj} = 0 \tag{2.11}$$

which contradict (2.8) again. Thus, the absence of relations (2.6) between the canonical variables is equivalent to the condition (2.8).

If Lagrangian (2.1) is non-singular, then the Euler equations (2.2) due to condition (2.8) can be represented in the normal form

$$\ddddot{x}_i = \ddddot{x}_i(x, \dot{x}, \ddot{x}, \dddot{x}) \qquad 1 \leqslant i \leqslant n. \tag{2.12}$$

As early as 1850 Ostrogradskii [19] showed that for non-degenerate Lagrangians a system of n equations of fourth order (2.2) or (2.12) is equivalent to a canonical system of $4n$ equations of first order

$$\begin{aligned} \dot{q}_{1i} &= \frac{\partial H}{\partial p_{1i}} & \dot{q}_{2i} &= \frac{\partial H}{\partial p_{2i}} \\ \dot{p}_{1i} &= -\frac{\partial H}{\partial q_{1i}} & \dot{p}_{2i} &= -\frac{\partial H}{\partial q_{2i}} \end{aligned} \qquad 1 \leqslant i \leqslant n \tag{2.13}$$

where the Hamiltonian H is defined by

$$H = p_1 \dot{x} + p_2 \ddot{x} - L(x, \dot{x}, \ddot{x}). \tag{2.14}$$

It is important that H can be represented only as a function of the canonical variables q_1, q_2, p_1, p_2. Indeed, using (2.5) we get from (2.14)

$$\begin{aligned} \mathrm{d}H &= \mathrm{d}p_1\, \dot{x} + p_1\, \mathrm{d}\dot{x} + \mathrm{d}p_2\, \ddot{x} + p_2\, \mathrm{d}\ddot{x} - \frac{\partial L}{\partial x}\mathrm{d}x - \frac{\partial L}{\partial \dot{x}}\mathrm{d}\dot{x} - \frac{\partial L}{\partial \ddot{x}}\mathrm{d}\ddot{x} \\ &= -\frac{\partial L}{\partial x}\mathrm{d}q_1 + \left(p_1 - \frac{\partial L}{\partial \dot{x}}\right)\mathrm{d}q_2 + \dot{q}_1\, \mathrm{d}p_1 + \dot{q}_2\, \mathrm{d}p_2. \end{aligned} \tag{2.15}$$

Thus, $\mathrm{d}H$ depends only on the differentials of the canonical variables, this being right both for non-degenerate Lagrangians and for degenerate ones. In both cases we have

$$H = H(q_1, q_2, p_1, p_2) \tag{2.16}$$

$$\mathrm{d}H = \frac{\partial H}{\partial q_1}\mathrm{d}q_1 + \frac{\partial H}{\partial q_2}\mathrm{d}q_2 + \frac{\partial H}{\partial p_1}\mathrm{d}p_1 + \frac{\partial H}{\partial p_2}\mathrm{d}p_2. \tag{2.17}$$

Substituting $p_1-(\partial L/\partial \dot{x})$ into (2.15) according to (2.4) by $-\dot{p}_2$ and $\partial L/\partial x$ by virtue of the Euler equations (2.2) by $\dot{p}_1$, we equate the right-hand sides in (2.15) and (2.17):

$$-\dot{p}_1\,\mathrm{d}q_1-\dot{p}_2\,\mathrm{d}q_2+\dot{q}_1\,\mathrm{d}p_1+\dot{q}_2\,\mathrm{d}p_2 =\frac{\partial H}{\partial q_1}\mathrm{d}q_1+\frac{\partial H}{\partial q_2}\mathrm{d}q_2+\frac{\partial H}{\partial p_1}\mathrm{d}p_1+\frac{\partial H}{\partial p_2}\mathrm{d}p_2. \tag{2.18}$$

Now we get

$$-\left(\dot{p}_1+\frac{\partial H}{\partial q_1}\right)\mathrm{d}q_1-\left(\dot{p}_2+\frac{\partial H}{\partial q_2}\right)+\left(\dot{q}_1-\frac{\partial H}{\partial p_1}\right)\mathrm{d}p_1+\left(\dot{q}_2-\frac{\partial H}{\partial p_2}\right)\mathrm{d}p_2=0. \tag{2.19}$$

For non-singular Lagrangians the canonical variables q_1, q_2, p_1 and p_2 are independent and as a consequence their differentials are independent. This enables one to equate to zero the coefficients of each differential in (2.1) and to obtain the canonical equations (2.13). It was in this way that Ostrogradskii [19] obtained equations (2.13).

If the action corresponding to the Lagrangian (2.1) is invariant under transformation $t\to t+\varepsilon$, then according to the first Noether theorem [20] the quantity

$$E(x,\dot{x},\ddot{x},\dddot{x})=H(q_1=x,\,q_2=\dot{x},\,p_1=p_1(x,\dot{x},\ddot{x},\dddot{x}),\,p_2=p_2(x,\dot{x},\ddot{x})) \tag{2.20}$$

is conserved on solutions of the equations of motion (2.2). Therefore E can naturally be called the energy.

3. Constraints in the phase space and the generalised Hamiltonian equations of motion

Let the initial Lagrangian (2.1) be singular. We suppose that in the whole range of variables x, $\dot{x}$ and $\ddot{x}$ the condition

$$\operatorname{rank}\|\Lambda_{ij}\|=r=n-m_1 \tag{3.1}$$

is satisfied. In this case the Euler equations (2.2) represent a system of r equations of fourth order and $m_1=n-r$ equations containing no $\ddddot{x}$. These last m_1 equations will be called the Lagrangian constraints. They can be separated from system (2.2) in the following way. Let $\xi_i^a(x,\dot{x},\ddot{x})$, $a=1,\ldots,m_1$, $i=1,\ldots,n$ be eigenvectors of the matrix Λ defined by (2.9) with zero eigenvalues

$$\xi_i^a(x,\dot{x},\ddot{x})\Lambda_{ij}(x,\dot{x},\ddot{x})=0 \qquad 1\leqslant i,j\leqslant n,\ 1\leqslant a\leqslant m_1. \tag{3.2}$$

The number of such vectors due to (3.1) is m_1. Projecting the Euler equations (2.2) on these eigenvectors we get m_1 Lagrangian constraints

$$B_a(x,\dot{x},\ddot{x},\dddot{x})=\xi_i^a\left(\frac{\partial L}{\partial x_i}-\dot{p}_{1i}\right) \qquad 1\leqslant a\leqslant m_1. \tag{3.3}$$

We suppose that the system of equations (2.2) is consistent. It will be satisfied, for example, in the case when the Lagrangian constraints containing no $\ddddot{x}$ define the invariant submanifold for equations of fourth order in (2.2) [15].

Taking into account (3.1) one can immediately obtain m_1 constraints on q_1, q_2 and p_2. For this purpose relations (2.5) have to be solved for r variables $\ddot{x}$ in the form

$$\ddot{x}_\alpha=\ddot{x}_\alpha(q_1,q_2,p_{2\beta},\ddot{x}_{r+1},\ldots,\ddot{x}_n) \qquad 1\leqslant\alpha,\beta\leqslant r. \tag{3.4}$$

Here we suppose that the first r rows and r columns of Λ are linearly independent. This can obviously be done always by a corresponding change of numeration of the variables x_i, $i=1,\ldots,n$. Substituting (3.2) into the remaining m_1 relations (2.5) we get m_1 constraints in the form

$$p_{2,r+a}=p_{2,r+a}(q_1,q_2,p_{2\beta}) \qquad a=1,\ldots,m_1=n-r \qquad \beta=1,\ldots,r. \tag{3.5}$$

These constraints or the set of constraints equivalent to them will be written hereafter in the following way:

$$\varphi_a(q_1,q_2,p_2)=0 \qquad a=1,\ldots,m_1. \tag{3.6}$$

Constraints (3.5) or (3.6), by analogy with the Dirac generalised Hamiltonian dynamics for singular Lagrangians without higher derivatives [11-14], can naturally be called the primary constraints, as they are a consequence of the singularity condition (3.1) for Lagrangian (2.1) and the definition of canonical momenta (2.5) without using the equations of motion (2.2). After substitution of the definitions (2.3) and (2.5) into constraints (3.6) the latter transform into m_1 identities for x, $\dot{x}$, $\ddot{x}$.

Replacing f in (2.11) by the primary constraints (3.6) one verifies that zero eigenvectors $\xi_i^a(x,\dot{x},\ddot{x})$, $1\leq a\leq m_1$, $1\leq i\leq n$, of the matrix Λ can always be chosen so that they transform by virtue of the definition (2.5) into the functions which depend only on the canonical variables q_1, q_2 and p_2, i.e. the dependence of $\ddot{x}$ disappears. Without loss of generality one can put

$$\xi_r^a(q_1,q_2,p_2)=\frac{\partial\varphi_a(q_1,q_2,p_2)}{\partial p_{2i}} \qquad 1\leq a\leq m_1,\ 1\leq i\leq n. \tag{3.7}$$

Let us try to transform the Euler equations (2.2) for singular Lagrangians into the phase space. For this purpose we replace the canonical momenta p_2 by their expressions in terms of q_1, q_2 and $\dot{q}_2$ according to (2.5) on the left- and right-hand sides of the definition of the canonical Hamiltonian

$$H(q_1,q_2,p_1,p_2)=p_1\dot{q}_1+p_2\dot{q}_2-L(q_1,q_2,\dot{q}_2). \tag{3.8}$$

As a result, we obtain an identity with respect to q_1, q_2, p_1 and $\dot{q}_2$. Differentiation of this identity with respect to $\dot{q}_2$ gives

$$\left(\frac{\overline{\partial H}}{\partial p_{2j}}-\dot{q}_{2j}\right)\frac{\partial\bar{p}_{2j}}{\partial\dot{q}_{2i}}=0 \qquad 1\leq i,j\leq n. \tag{3.9}$$

The bar means the replacement described above, i.e.

$$\bar{f}(q_1,q_2,p_1,p_2)\equiv f\left(q_1,q_2,p_1,\frac{\partial L(q_1,q_2,\dot{q}_2)}{\partial\dot{q}_2}\right)\equiv F(q_1,q_2,p_1,\dot{q}_2). \tag{3.10}$$

Since $\partial\bar{p}_{2j}/\partial\dot{q}_{2i}=\Lambda_{ij}(q_1,q_2,\dot{q}_2)$, it follows from (3.9) that the quantities

$$\dot{q}_{2j}-\frac{\overline{\partial H}}{\partial p_{2j}} \qquad 1\leq j\leq n \tag{3.11}$$

are eigenvectors of the matrix $\Lambda(q_1,q_2,\dot{q}_2)$ with zero eigenvalue. This vector can be decomposed over a complete set of zero eigenvectors of the matrix Λ:

$$\dot{q}_{2j} - \overline{\frac{\partial H}{\partial p_{2j}}} = \sum_{a=1}^{m_1} \lambda_a(q_1, q_2, p_1, \dot{q}_2)\xi_j^a(q_1, q_2, \dot{q}_2)$$

$$= \sum_{a=1}^{m_1} \lambda_a(q_1, q_2, p_1, \dot{q}_2) \frac{\overline{\partial \varphi_a(q_1, q_2, p_2)}}{\partial p_{2j}}. \qquad (3.12)$$

Here we have used (3.7).

Let us substitute (2.5) into (3.8), differentiate the identity obtained with respect to q_2, and take into account the relation

$$p_1 + \dot{p}_2 = \partial L/\partial q_2 \qquad (3.13)$$

which follows from (2.4) and (2.5). As a result, we obtain

$$\overline{\frac{\partial H}{\partial q_{2i}}} + \dot{p}_{2i} = \left(\dot{q}_{2j} - \overline{\frac{\partial H}{\partial p_{2j}}}\right) \frac{\partial \bar{p}_{2j}}{\partial q_{2i}} = \sum_{a=1}^{m_1} \lambda_a(q_1, q_2, p_1, \dot{q}_2) = \overline{\frac{\partial \varphi_a}{\partial p_{2j}}} \frac{\partial \bar{p}_{2j}}{\partial q_{2i}}. \qquad (3.14)$$

Differentiation with respect to q_1 and q_2 of the identities, which appear upon transforming the primary constraints (3.6) by substitution of (2.5) into them, gives

$$\overline{\frac{\partial \varphi_a}{\partial q_{si}}} = -\overline{\frac{\partial \varphi_a}{\partial p_{2j}}} \frac{\partial \bar{p}_{2j}}{\partial q_{si}} \qquad s = 1, 2 \quad 1 \leqslant i, j \leqslant n. \qquad (3.15)$$

Now (3.12) can be rewritten in the form

$$\dot{p}_{2i} - \overline{\frac{\partial H}{\partial q_{2i}}} = -\sum_{a=1}^{m_1} \lambda_a(q_1, q_2, p_1, \dot{q}_2) \overline{\frac{\partial \varphi_a}{\partial q_{2i}}} \qquad 1 \leqslant i \leqslant n. \qquad (3.16a)$$

Taking into account that the Euler equations (2.2) can be cast in the form

$$\dot{p}_1 = \partial L/\partial q_1 \qquad (3.16b)$$

we obtain

$$\dot{p}_{1i} + \overline{\frac{\partial H}{\partial q_{1i}}} = -\sum_{a=1}^{m_1} \lambda_a(q_1, q_2, p_1, \dot{q}_2) \overline{\frac{\partial \varphi_a}{\partial q_{1i}}} \qquad 1 \leqslant i \leqslant n. \qquad (3.17)$$

Finally differentiation of (3.8) with respect to p_1 gives

$$\dot{q}_{1i} - \overline{\frac{\partial H}{\partial p_{1i}}} = 0 \qquad 1 \leqslant i \leqslant n. \qquad (3.18)$$

We now introduce the Poisson brackets in the usual way:

$$(f, g) = \frac{\partial f}{\partial q_{si}} \frac{\partial g}{\partial p_{si}} - \frac{\partial f}{\partial p_{si}} \frac{\partial g}{\partial q_{si}} \qquad s = 1, 2 \quad i = 1, \ldots, n$$

$$f = f(q_1, q_2, p_1, p_2) \qquad g = g(q_1, q_2, p_1, p_2). \qquad (3.19)$$

Using them we can write (3.12), (3.16*a*), (3.17) and (3.18) in the form

$$\dot{z} = \overline{(z, H)} + \sum_{a=1}^{m_1} \lambda_a(q_1, q_2, p_1, \dot{q}_2)\overline{(z, \varphi_a)}. \qquad (3.20)$$

Here z means a complete set of the canonical variables q_1, q_2, p_1 and p_2.

We recall that equations (3.20) are written in terms of the variables q_1, q_2, p_1 and $\dot{q}_2$. The expressions $\overline{(z, H)}$ and $\overline{(z, \varphi_a)}$ can obviously be transformed into the phase space if we take account of (2.5). As a result, we get the functions of the canonical variables (z, H) and (z, φ_a) respectively. The correspondence between $\overline{(z, H)}$, $\overline{(z, \varphi_a)}$ and (z, H), (z, φ_a) is one-to-one only on the submanifold of the phase space defined by the primary constraints (3.5) or (3.6). The dependence on $\dot{q}_2$ in the functions $\lambda_a(q_1, q_2, p_1, \dot{q}_2)$ does not disappear by virtue of (2.5). In order to prove this, it is sufficient to act on the left- and right-hand sides of (3.12) by the following linear differential operators [21]:

$$X^a = \xi^a_j \frac{\partial}{\partial \dot{q}_{2j}} \qquad a = 1, 2, \ldots, m_1. \tag{3.21}$$

This gives

$$X^a \lambda_b(q_1, q_2, p_1, \dot{q}_2) = \delta_{ab} \neq 0. \tag{3.22}$$

If one takes the primary constraints in the resolved form (3.5), then the functions λ_a reduce in this case to $\dot{q}_{2,r+a}$, $a = 1, \ldots, m_1$.

Thus, the only way to transform equations (3.20) into the phase space is to eliminate the functions $\lambda_a(q_1, q_2, p_1, \dot{q}_2)$ imposing additional conditions on the solutions of these equations†. From this point on, we are actually dealing with the Dirac system with primary constraints [13].

In the Dirac approach, however, the equations of motion in the phase space were obtained by the Lagrangian method of indefinite multipliers. Therefore the functions λ_a were considered at first as unknown functions of time determined by additional conditions on the solutions of the equations of motion. One demands that the time derivatives of the primary constraints vanish on the solutions of these equations. As is known, all the secondary constraints can be obtained in this way and some number of functions λ_a can be expressed in terms of the canonical variables. The remaining undetermined functions $\lambda_a(t)$, the number of which equals the number of the primary first-class constraints, describe the functional freedom in the theory. But in the Dirac reasoning there are no convincing arguments why it is sufficient to take into account only the primary constraints in order to obtain the equations of motion in the phase space by the Lagrangian method of indefinite multipliers. In our opinion, the derivation of these equations by the differentiation of the canonical Hamiltonian fills this gap. Another method of obtaining the equations of motion in the phase space for singular Lagrangians of arbitrary order, which avoids this problem, is developed in [16].

So, we shall further follow the Dirac reasoning. Let us demand that the time derivatives of the primary constraints vanish on the solutions of equations (3.20)

$$\begin{gathered} \frac{\mathrm{d}\bar{\varphi}_a}{\mathrm{d}t} = \overline{(\varphi_a, H)} + \sum_{b=1}^{m_1} \lambda_b(q_1, q_2, p_1, \dot{q}_2)\overline{(\varphi_a, \varphi_b)} \overset{\bar{\varphi}_c}{\simeq} 0 \\ a, b, c = 1, \ldots, m_1. \end{gathered} \tag{3.23}$$

Here the sign $\overset{\bar{\varphi}_c}{\simeq}$ means a weak equality when the conditions $\bar{\varphi}_c = 0$ are satisfied. The expressions $\overline{(\varphi_a, H)}$ and $\overline{(\varphi_a, \varphi_b)}$ can be transformed into the phase space if we take into account (2.5). Hence one can express from (3.23) r_1 functions λ_a in terms of the

† If the Lagrangian L is non-degenerate, i.e. rank $\Lambda = n$, then it follows from (3.9) that (3.11) vanishes and on the right-hand sides of (3.16), (3.17) and (3.18) we have zeros. As a result, we get the canonical Ostrogradskii equations (2.13).

canonical variables where

$$r_1 = \operatorname{rank}\|\overline{(\varphi_a, \varphi_b)}\| = \operatorname{rank}\|(\varphi_a, \varphi_b)\|\|_{\varphi_c=0}. \tag{3.24}$$

The remaining $\mu_1 = m_1 - r_1$ equations in (3.21) give rise to μ_1 constraints on the canonical variables

$$\omega_{s_1}(q_1, q_2, p_1, p_2) = 0 \qquad s_1 = 1, 2, \ldots, \mu_1. \tag{3.25}$$

It is obvious how to change the considerations when some or all of equations (3.23) are satisfied identically. Further it is necessary to demand that

$$\frac{\mathrm{d}\bar{\omega}_{s_1}}{\mathrm{d}t} \overset{\varphi,\omega}{\cong} 0 \qquad s_1 = 1, \ldots, \mu_1 \tag{3.26}$$

and so on. As a result, all the secondary constraints can be obtained in this way and m functions $\lambda_a(q_1, q_2, p_1, \dot{q}_2)$ remain undetermined in terms of the canonical variables, where m is the number of primary first-class constraints. The theory does not enable us to fix them, and they remain absolutely arbitrary functions of their arguments. Therefore one can consider them as arbitrary functions of time. As a result, equations (3.20) prove to be transformed into the phase space completely.

In order to get a correct final result one might originally have considered the functions $\lambda_a(q_1, q_2, p_1, \dot{q}_2)$ in (3.20) as unknown functions of time. This would have enabled us to go in the phase space immediately

$$\dot{z} = (z, H) + \sum_{a=1}^{m_1} \lambda_a(t)(z, \varphi_a). \tag{3.27}$$

The initial consideration of equations (3.20) in terms of the variables $q_1, q_2, p_1, \dot{q}_2$, as given above, justifies this procedure.

4. Derivation of the secondary constraints in the framework of the Lagrangian formalism

In the preceding section the secondary constraints were obtained by successive differentiation with respect to time of the primary constraints using the equations of motion in form (3.20) or (3.27). For this purpose, however, one can use the Euler equations in form (3.16*a*) and, as in the case of a singular Lagrangian of the first order, this approach enables us to obtain some additional information about the secondary constraints [21] and trace the relation between the Lagrangian and Hamiltonian descriptions [21-23].

Differentiation with respect to time of the left-hand sides in equations of primary constraints (3.6) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\bar{\varphi}_a(q_1, q_2, p_2) = \overline{\frac{\partial \varphi_a}{\partial q_{1i}}}\,\dot{q}_{1i} + \overline{\frac{\partial \varphi_a}{\partial q_{2i}}}\,\dot{q}_{2i} + \overline{\frac{\partial \varphi_a}{\partial p_{2i}}}\,\overline{\dot{p}_{2i}} \qquad a = 1, \ldots, m_1. \tag{4.1}$$

Now we replace the derivatives with respect to the coordinates q_1 and q_2 in (4.1) according to (3.19) and take into account (3.13). As a result, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\bar{\varphi}_a(q_1, q_2, p_2) = -\overline{\frac{\partial \varphi_a}{\partial p_{2j}}}\left(\frac{\partial^2 L}{\partial \ddot{x}_j \partial x_i}\,\dot{x}_i + \frac{\partial^2 L}{\partial \ddot{x}_j \partial \dot{x}_i}\,\ddot{x}_i - \frac{\partial L}{\partial \dot{x}_j} + p_{1j}\right) = 0 \qquad a = 1, \ldots, m_1. \tag{4.2}$$

The expression in parentheses vanishes due to (2.4). Thus the derivative $(\mathrm{d}/\mathrm{d}t)\bar{\varphi}_a(q_1, q_2, p_2)$ is equal to zero without using the equations of motion. In addition the questions

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\varphi}_a(q_1, q_2, p_2) = 0 \qquad 1 \leqslant a \leqslant m_1 \tag{4.3}$$

are equivalent to the following relations:

$$\xi_i^a(q_1, q_2, \dot{q}_2)\bar{p}_{1i} = \xi_i^a(q_1, q_2, \dot{q}_2)\left(\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial^2 L}{\partial \ddot{x}_i \partial x_j}\dot{x}_j - \frac{\partial^2 L}{\partial \ddot{x}_i \partial \dot{x}_j}\ddot{x}_j\right) \qquad a = 1, \ldots, m_1. \tag{4.4}$$

Let us now investigate the question: what are the conditions under which equations (4.4) transform due to the definitions (2.3)-(2.5) into equations containing only the canonical variables q_1, q_2, p_1, p_2 and give, as a result, the secondary Hamiltonian constraints. For this purpose one has to act on the right-hand side of (4.5) by the operators (3.21). This gives [21]

$$\xi_i^a\,\xi_j^b\left(\frac{\partial^2 L}{\partial \dot{x}_i \partial \ddot{x}_j} - \frac{\partial^2 L}{\partial \ddot{x}_i \partial \dot{x}_j}\right) = \overline{(\varphi_a, \varphi_b)} \qquad a, b, c = 1, \ldots, m_1. \tag{4.5}$$

Hence, if there are primary constraints which are in involution, at least in a weak sense, with the whole set of the primary constraints (3.6), then for the corresponding values of the index a in (4.4) the action of the operators (3.21) on the right-hand side of (4.4) gives zero. In this case the variables $\ddot{x}$ on the right-hand side of (4.4) can be eliminated by virtue of (2.5) and equations (4.4) give us the secondary constraints on the canonical variables. The number of these constraints is equal to the number of primary constraints which are in involution, at least in a weak sense, with the whole set of the primary constraints (3.6). Obviously, these constraints are the same secondary constraints (3.25) obtained in the preceding section by the Dirac method. From (4.4) it follows immediately that these constraints are linear in p_1 and they are obtained by projection of the definition (2.4) on the zero eigenvectors of the matrix Λ.

Further one must differentiate the constraints (3.25) with respect to time

$$\frac{\mathrm{d}\omega_{s_1}}{\mathrm{d}t} = \frac{\partial \omega_{s_1}}{\partial q_1}\dot{q}_1 + \frac{\partial \omega_{s_1}}{\partial q_2}\dot{q}_2 + \frac{\partial \omega_{s_1}}{\partial p_1}\dot{p}_1 + \frac{\partial \omega_{s_1}}{\partial p_2}\dot{p}_2 = 0 \qquad s_1 = 1, \ldots, \mu_1 \tag{4.6}$$

and use (3.13) and equations of motion in the form (3.16a). If using (2.5) we can eliminate $\ddot{x}$ from all the equations (4.6) or from some of them, then we obtain some more secondary constraints

$$\omega_{s_2}(q_1, q_2, p_1, p_2) = 0 \qquad s_2 = \mu_1 + 1, \ldots, \mu_2. \tag{4.7}$$

This procedure of successive differentiation of the constraints must be continued until the appearance of the new constraints stops or the variables $\ddot{x}$ cannot be eliminated from all the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_{s_{k+1}}(q_1, q_2, p_1, p_2) = 0 \qquad s_{k+1} = \mu_k + 1, \ldots \tag{4.8}$$

using the definition (2.5). As a result, all the secondary constraints will be obtained

$$\omega_s(q_1, q_2, p_1, p_2) = 0 \qquad s = 1, \ldots, m_2 \qquad m_2 = \mu_1 + \mu_2 + \ldots + \mu_k. \tag{4.9}$$

Let us establish the relation between Hamiltonian and Lagrangian constraints. First of all we show that the differentiation with respect to time of equations (4.5), which leads to the first set of the secondary constraints (3.25) gives, by virtue of the equations of motion (2.2), the Lagrangian constraints (3.3). Equations (4.5) can be represented in the form

$$\xi_i^a\left(p_{1i}-\frac{\partial L}{\partial \dot{x}_i}+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{x}_i}\right)=0 \qquad a=1,\ldots,m_1. \tag{4.10}$$

The differentiation with respect to time of the left-hand sides of these equalities gives

$$\xi_i^a\left(\dot{p}_{1i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_i}+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial \ddot{x}_i}\right)+\left(\frac{\mathrm{d}}{\mathrm{d}t}\xi_i^a\right)\left(p_{1i}-\frac{\partial L}{\partial \dot{x}_i}+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{x}_i}\right)=0. \tag{4.11}$$

In the first term in (4.11) we make the following substitution using equations of motion (2.2)

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_i}+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial \ddot{x}_i}=-\frac{\partial L}{\partial x_i}. \tag{4.12}$$

The second term in (4.11) vanishes due to the definition (2.4). As a result, from (4.11) we get the Lagrangian constraints (3.3).

The procedure of differentiation of the Lagrangian constraints with respect to time is important for the Lagrangian formalism too. It is in fact the search of the invariant submanifold in the space with the coordinates x, $\dot{x}$, $\ddot{x}$, $\dddot{x}$. The Cauchy data for the Euler equations (2.2) must belong to this submanifold. Only for this constraint set of the initial data can one consistently formulate the Cauchy problem for equations (2.2).

It is clear by the construction that for the primary constraints (3.6) and for the first set of the secondary ones (3.25) there are no corresponding Lagrangian constraints, as the substitution of (2.4) and (2.5) into (3.6) and (3.25) gives the identities.

5. The generalisation of the relativistic point action

As an example, we consider the following generalisation of the point particle action [18]:

$$S=-m\int \mathrm{d}s+\alpha\int k\,\mathrm{d}s \tag{5.1}$$

where m is a parameter with dimension of mass, α is a dimensionless constant, $\mathrm{d}s$ is the differential of the particle world trajectory $\mathrm{d}s^2=\mathrm{d}x_\mu\,\mathrm{d}x^\mu$, k is the curvature of this trajectory $k^2=(\mathrm{d}^2x/\mathrm{d}s^2)^2$. With a given parametrisation $x^\mu(\tau)$, $\mu=0,1,2,\ldots,D-1$, action (4.1) is rewritten in the form

$$S=-m\int\sqrt{\dot{x}^2}\,\mathrm{d}\tau+\alpha\int\frac{((\dot{x}\ddot{x})^2-\dot{x}^2\ddot{x}^2)^{1/2}}{\dot{x}^2}\,\mathrm{d}\tau \qquad \dot{x}=\frac{\mathrm{d}x}{\mathrm{d}\tau}. \tag{5.2}$$

The metric with the signature $\eta_{\mu\nu}=\mathrm{diag}\,(+,-,-,\ldots)$ is used.

The matrix Λ defined in (2.9) is given in the case under consideration by

$$\Lambda_{\mu\nu}=\frac{\alpha}{\dot{x}^2\sqrt{g}}\left(\dot{x}_\mu\dot{x}_\nu-\dot{x}^2\eta_{\mu\nu}-\frac{l_\mu l_\nu}{g}\right) \tag{5.3}$$

where

$$\begin{aligned} &l_\mu=(\dot{x}\ddot{x})\dot{x}_\mu-\dot{x}^2\ddot{x}_\mu, \qquad g=(\dot{x}\ddot{x})^2-\dot{x}^2\ddot{x}^2\\ &\dot{x}^\mu l_\mu=0 \qquad l_\mu l^\mu=-g\dot{x}^2. \end{aligned} \tag{5.4}$$

It is then easy to be convinced that the matrix Λ has two eigenvectors with zero eigenvalues $\dot{x}^{\mu}$ and l^{μ}. Hence, the theory must involve two primary constraints.

Using the definition†

$$p_{2\mu}=-\frac{\partial L}{\partial \ddot{x}^{\mu}}=-\frac{\alpha}{\dot{x}^{2}}\frac{l_{\mu}}{\sqrt{g}} \tag{5.5}$$

and equations (4.4) we obtain the primary constraints corresponding to (3.4):

$$\varphi_{1}=p_{2}q_{2}=0 \tag{5.6}$$

$$\varphi_{2}=p_{2}^{2}q_{2}^{2}+\alpha^{2}=0 \tag{5.7}$$

where $q_{2\mu}=\dot{x}_{\mu}$.

We get the secondary constraints in the model under consideration at first by a method described in § 4. The Poisson brackets will be defined as follows:

$$(f, g)=\sum_{s=1}^{2}\left(\frac{\partial f}{\partial p_{s}^{\mu}}\frac{\partial g}{\partial q_{s\mu}}-\frac{\partial f}{\partial q_{s}^{\mu}}\frac{\partial g}{\partial p_{s\mu}}\right). \tag{5.8}$$

The primary constraints (5.6) and (5.7) are in involution between themselves in a strong sense $(\varphi_{1}, \varphi_{2})=0$. Therefore there must be two secondary constraints which can be obtained by projection of the definition

$$p_{1\mu}=-\frac{\partial L}{\partial \dot{x}^{\mu}}+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{x}^{\mu}}=-\frac{\partial L}{\partial \dot{x}^{\mu}}-\dot{p}_{2\mu} \tag{5.9}$$

on the zero eigenvectors of the matrix $\Lambda: \xi_{\mu}^{1}=\dot{x}_{\mu}=q_{2\mu}$, $\xi_{\mu}^{2}=l_{\mu}\sim p_{2\mu}$. Projection of $q_{2\mu}$ on (5.9) gives

$$\omega_{1}=p_{1}q_{2}-m\sqrt{q_{2}^{2}}=0. \tag{5.10}$$

Finally, multiplying (5.9) by $p_{2\mu}$ we obtain

$$\omega_{2}=p_{1}p_{2}=0. \tag{5.11}$$

Differentiation with respect to time of (5.10) does not give new constraints. Differentiating (5.11) with respect to time and taking into account the equations of motion

$$\dot{p}_{1}=0 \tag{5.12}$$

and constraints (5.6)-(5.10) we obtain the expression

$$\frac{\mathrm{d}\omega_{2}}{\mathrm{d}t}=p_{1}\dot{p}_{2}=-p_{1}\left(p_{1}+\frac{\partial L}{\partial \dot{x}}\right)=-p_{1}^{2}+m^{2}+\frac{\alpha}{\sqrt{q_{2}^{2}}}p_{1}q_{2}\sqrt{g}. \tag{5.13}$$

One cannot eliminate $\ddot{x}$ from (5.13) using (5.5). Indeed

$$\ddot{x}^{\mu}\frac{\partial g}{\partial \ddot{x}^{\mu}}=2g\neq 0. \tag{5.14}$$

Thus the constraints (5.6), (5.7), (5.10) and (5.11) exhaust the whole set of constraints in the model under consideration. In contrast to the conclusion in [18, 24] we have here four constraints.

† The minus sign is introduced in order to get (2.5) for the spacelike components of p_{2}.

It follows from definition (5.5) that

$$p_2\ddot{x} = -\frac{\alpha}{\dot{x}^2}\sqrt{g}. \tag{5.14'}$$

Therefore we get the following expression for the canonical Hamiltonian†:

$$H = -p_1\dot{x} - p_2\ddot{x} - L = -p_1q_2 + m\sqrt{q_2^2} = -\omega_1. \tag{5.15}$$

Let us evaluate the Poisson brackets between all the constraints and construct the matrix Δ

$$\begin{gathered}\Delta_{AB} = (\theta_A, \theta_B) \qquad 1 \leqslant A, B \leqslant 4 \\ \theta_1 = \varphi_1 \qquad \theta_2 = \varphi_2 \qquad \theta_3 = \omega_1 \qquad \theta_4 = \omega_2.\end{gathered} \tag{5.16}$$

On the submanifold M of the phase space defined by the constraint equations

$$\theta_A(q_1, q_2, p_1, p_2) = 0 \qquad A = 1, \ldots, 4 \tag{5.17}$$

the following elements of the matrix Δ are different from zero:

$$\begin{gathered}\Delta_{24} = (\theta_2, \theta_4) = (\varphi_2, \omega_2) = -2p_2^2(p_1q_2) \\ \Delta_{34} = (\theta_3, \theta_4) = (\omega_1, \omega_2) = m^2 - p_1^2.\end{gathered} \tag{5.18}$$

Thus, we have on M rank $\Delta = 2$. Hence there are two first-class constraints and two second-class constraints in this theory. Let us pick out these constraints explicitly. For this purpose we go to the equivalent set of constraints [15]

$$\begin{gathered}\phi_s = \xi_A^s\,\theta_A \qquad s = 1, 2 \\ \phi_3 = \theta_3 = \omega_1 \qquad \phi_4 = \theta_4 = \omega_2\end{gathered} \tag{5.19}$$

where ξ_A^s, $s = 1, 2$, $A = 1, \ldots, 4$ are two zero eigenvectors of the matrix Δ. These vectors can be taken in the following form:

$$\begin{gathered}\xi_1^1 = 1 \qquad \xi_2^1 = \xi_3^1 = \xi_4^1 = 0 \\ \xi_1^2 = 0 \qquad \xi_2^2 = m^2 - p_1^2 \qquad \xi_3^2 = 2p_2^2(p_1q_2) \qquad \xi_4^2 = 0.\end{gathered} \tag{5.20}$$

As a result, we get the new set of constraints

$$\begin{gathered}\phi_1 = p_2q_2 = 0 \\ \phi_2 = (m^2 - p_1^2)(p_2^2q_2^2 + \alpha^2) + 2p_2^2(p_1q_2)(p_1q_2 - m\sqrt{q_2^2}) = 0, \\ \phi_3 = p_1q_2 - m\sqrt{q_2^2} = 0 \qquad \phi_4 = p_1p_2 = 0\end{gathered} \tag{5.21}$$

which are equivalent to the initial constraints $\theta_A = 0$, $A = 1, \ldots, 4$. This means that equations (5.21) define the same submanifold M in the phase space. But for constraints ϕ_A, $A = 1, \ldots, 4$, there is only one Poisson bracket different from zero on M:

$$(\phi_3, \phi_4) = m^2 - p_1^2.$$

Thus, the constraints ϕ_1 and ϕ_2 are the first-class constraints.

† If we substitute in $H(q_1, q_2, p_1, p_2)$ the canonical momenta p_1 and p_2 by their expressions in terms of $\dot{x}$, $\ddot{x}$, $\dddot{x}$ according to (5.5) and (5.9) we get zero identically. It is the consequence of the invariance of the action (5.2) under the transformation $\tilde{\tau} = f(\tau)$ with the arbitrary function f.

It is interesting to note that in the phase space there is the invariant submanifold defined by the constraints (5.21) and by the equation

$$\phi_5 = p_1^2 - m^2 = 0 \qquad (\phi_\alpha, \phi_\beta) \approx 0 \qquad \alpha, \beta = 1, \ldots, 5.$$

Let us now obtain the secondary constraints in this model by the Dirac method. Taking into account (5.15) we get

$$(\varphi_1, H) + \sum_{a=1}^{2} \lambda_a (\varphi_1, \varphi_a) = (\varphi_1, H) = -\omega_1 = 0$$

$$\begin{aligned}(\varphi_2, H) &+ \sum_{a=1}^{2} \lambda_a (\varphi_2, \varphi_a) \\ &= (\varphi_2, H) = -2(p_1 p_2) q_2^2 + 2q_2^2 \frac{(p_2 q_2)}{\sqrt{q_2^2}} \overset{\varphi_1}{\simeq} -2(p_1 p_2) q_2^2 \\ &= -2q_2^2 \omega_2 = 0.\end{aligned}$$

The requirement of the stationarity of the secondary constraints ω_1 and ω_2 enables us to express λ_2 in terms of the canonical variables

$$\lambda_2 = \frac{p_1^2 - m^2}{2p_2^2 (p_1 q_2)}.$$

The Hamiltonian which defines the dynamics in the phase space is

$$H_{\mathrm{T}} = H + \lambda_1(\tau) \varphi_1 + \frac{p_1^2 - m^2}{2p_2^2 (p_1 q_2)} \varphi_2.$$

The mass of the system described by the action (5.2) should be defined as

$$M^2 = p_1^2. \tag{5.22}$$

Indeed, p_1^μ is the conserved Noether vector of the energy-momentum in this theory. The variation of the action (5.2) can be written in the form

$$\delta S = -[p_{1\mu} \delta x^\mu + p_{2\mu} \delta \dot{x}^\mu]|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \frac{\delta S}{\delta x^\mu} \delta x^\mu.$$

If the equations of motion $\delta S / \delta x^\mu = 0$ are satisfied and $\delta x^\mu = \text{constant}$, then $p_{1\mu}(\tau_1) = p_{1\mu}(\tau_2)$.

Let us show now that the constraints (5.6), (5.7), (5.10) and (5.11) do not lead to M^2 in (5.22) being positive. For this purpose it is convenient to use the accompanying Lorentz frame where $q_2^\mu = \dot{x}^\mu = (q_2^0, 0, 0, \ldots)$. To remove the superlight velocities, one should impose the condition $q_2^2 > 0$. This condition is quite accessible from the standpoint of the generalised Hamiltonian dynamics, since in the theory under consideration there are two first-class constraints ϕ_1 and ϕ_2 in (5.21). As a consequence, one should impose two gauge conditions on the canonical variables. The proper time gauge

$$q_2^2 = \dot{x}^2 = \text{constant} > 0 \tag{5.23}$$

is obviously suitable as one of them.

In the accompanying reference frame introduced above we get from (5.6) $p_2^0 = 0$. The constraint (5.7) reduce to $\boldsymbol{p}_2^2 = \alpha^2/(q_2^0)^2$ and from (5.10) we obtain $(p_1^0)^2 = m^2$. Thus, in this reference frame we can write

$$M^2 = p_1^2 = (p_1^0)^2 - \boldsymbol{p}_1^2 = m^2 - \boldsymbol{p}_1^2. \tag{5.24}$$

The spacelike vector $p_1(\tau)$ is arbitrary except for the condition $p_1 \cdot p_2 = 0$ following from (5.11). Choosing the corresponding initial data for $p_1(\tau)$ we can obtain an arbitrary sign for M^2 in (5.24). One could suppose that the positive sign of M^2 may be provided by the second gauge condition. Unfortunately (5.22) cannot be used in this way. This condition must involve p_2, q_2 or q_1 without fail.

The indefinite sign of M^2 at the classical level should reveal itself in direct consequence in quantum theory [25].

6. Conclusion

The method proposed here enables one to construct the Hamiltonian formalism for systems described by singular Lagrangians of the second order. Obviously, the generalisation of this procedure to singular Lagrangians containing the derivatives of higher order meets no significant difficulties.

It would be interesting to make clear the connection of the invariance properties of the initial degenerate action with the number of the Hamiltonian constraints in the theory and with the properties of their Poisson brackets.

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